# Anaphoric Operators and Expressivity in Intensional Languages

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Logik Kolloquium Universität Konstanz, November 19, 2018

## 1 Introduction

### 1.1 Examples

(1) $\Diamond\Diamond(\varphi \wedge \rotatebox[origin=c]{180}{\textbf{R}}_1\psi)$

(2) $\Diamond(\psi \wedge \Diamond\varphi)$

(3) $\Diamond(\forall x)[\rotatebox[origin=c]{180}{\textbf{R}}_0(Q(x)) \to Q(x)]$

(4) Every man who ever$_1$ supported the Vietnam War will$_2$ have to admit$^1$ that now$_c$ he believes$^2$ that he was an idiot then$_1$. (Saarinen, 1979, 343)

(5) a. $(\forall x)[M_{w,\mathbf{t}}(x) \to (\forall \mathbf{t}' < \mathbf{t})[S_{w,\mathbf{t}'}(x) \to (\exists \mathbf{t}'' > \mathbf{t})$
$(\forall w')[A_{w,\mathbf{t}''}(x)(w') \to (\forall w'')[B_{w',\mathbf{t}_c}(x)(w'') \to I_{w'',\mathbf{t}'}(x)]]]]$

b. $(\forall x)[M_{w,\mathbf{t}}(x) \to (\forall \mathbf{t}' < \mathbf{t})[S_{w,\mathbf{t}'}(x) \to (\exists \mathbf{t}'' > \mathbf{t}')$
$(\forall w')[A_{w,\mathbf{t}''}(x)(w') \to (\forall w'')[B_{w',\mathbf{t}_c}(x)(w'') \to I_{w'',\mathbf{t}'}(x)]]]]$

c. $(\forall x)[M_{w,\mathbf{t}}(x) \to (\forall \mathbf{t}' < \mathbf{t})[S_{w,\mathbf{t}'}(x) \to (\exists \mathbf{t}'' > \mathbf{t}_c)$
$(\forall w')[A_{w,\mathbf{t}''}(x)(w') \to (\forall w'')[B_{w',\mathbf{t}_c}(x)(w'') \to I_{w'',\mathbf{t}'}(x)]]]]$

### 1.2 Setting the stage

- Relevant factors when it comes to comparing expressivity:

(a) the formal languages to which the formulae compared belong;

(b) the choice of the determinants of denotation;

(c) the relevant notions of denotation to be preserved in that comparison.

(6) $\Box\Diamond p \to \Diamond\Box p$

(7) $[(\forall i_1)[i_0Ri_1 \to (\exists i_2)[i_1Ri_2 \wedge P(i_2)]] \to$
$(\exists i_1)[i_0Ri_1 \wedge (\forall i_2)[i_1Ri_2 \to P(i_2)]]]$

(8) $(\forall i_0)\ i_0Ri_0$

(9) $(\forall p)[\Box p \vee \Box\ \neg p]$

(10) $(\forall P)[(\forall i_0)P(i_0) \vee (\forall i_0)\neg P(i_0)]$

(11) $\varphi$ is *true throughout* a frame $(W, \mathrm{R})$ iff $\varphi$ is true in all models $(W, \mathrm{R}, V, w)$.

(12) $\varphi$ is *true in* a context $c$ iff $[\![\varphi]\!]^{M,c,i_c} = 1$.

(13) $[\![\mathbf{P}\varphi]\!]^{(w,t,\dots),g} = 1$ iff $[\![\varphi]\!]^{(w,t',\dots),g}$, for some $t' < t$

(14) a. $(\exists t_1)[t_0 S t_1 \wedge \varphi[{}^{t_0}/_{t_1}]]$ where $[\![xSy]\!]^g = 1$ iff $g(y) < g(x)$

b. $(\exists i_1)[i_0 R i_1 \wedge \varphi[{}^{i_0}/_{i_1}]]$

where $[\![xRy]\!]^g = 1$ iff $g(x)_1 = g(y)_1$ and $g(y)_2 < g(x)_2$

(15) Every horse neighs.

(16) a. $(\forall x)[H(x) \to N(x)]$

b. $\lambda P.(\forall x)[H(x) \to P(x)]$

c. $\lambda Q.\lambda P.(\forall x)[Q(x) \to P(x)]$

(17) a. $[\lambda i_0.[\lambda i_1.[i_0 = i_1]]]$

b. $[{}^\wedge[\lambda F.[F = \lambda p.{}^\vee p]](\lambda p.{}^\vee p)]$

# 2 Intensional Type Logic

## 2.1 Syntax and Semantics of (Tensed) Intensional Type Logic **Montague (1970, 1973)**

(18) *Intensional Types*

Let $e$, $t$, and $s$ be some fixed distinct objects. Then $T_{IL}$ is the smallest set satisfying the following conditions:

a. $t \in T_{IL}$; $e \in T_{IL}$;

b. if $\sigma, \tau \in T_{IL}$, then $\langle \sigma, \tau \rangle \in T_{IL}$;

c. if $\tau \in T_{IL}$, then $\langle s, \tau \rangle \in T_{IL}$.

(19) Basic Terms of *IL*

a. $Con_\tau$ is the (denumerably infinite) set of all non-logical constants of type $\tau \in T_{IL}$;

b. $Var_\tau$ is the (denumerably infinite) set of all variables of type $\tau \in T_{IL}$.

(20) The family $(IL_\tau)_{\tau \in T_{IL}}$ of *IL*-terms

a. If $\mathbf{c} \in Con_\tau$, then $\mathbf{c} \in IL_\tau$;

b. If $x \in Var_\tau$, then $x \in IL_\tau$;

c. If $\alpha, \beta \in IL_\tau$, then $[\alpha = \beta] \in IL_t$;

d. If $x \in Var_\sigma$ and $\alpha \in IL_\tau$, then $[\lambda x.\alpha] \in IL_{\langle \sigma, \tau \rangle}$;

e. If $\alpha \in IL_{\langle \sigma, \tau \rangle}$ and $\beta \in IL_\sigma$, then $\alpha(\beta) \in IL_\tau$;

f. If $\alpha \in IL_\tau$, then $[{}^\wedge \alpha] \in IL_{\langle s, \tau \rangle}$;

g. If $\alpha \in IL_{\langle s, \tau \rangle}$ then $[{}^\vee \alpha] \in IL_\tau$.

(21) Domains

a. $D_e = D$, a (fixed and non-empty) set of (possible) individuals

b. $D_t = \{0, 1\}$

c. $D_{\langle \sigma, \tau \rangle} = D_\tau^{D_\sigma}$

d. $D_{\langle s, \tau \rangle} = D_\tau^{(W \times T)}$, where $W$ and $T$ are (fixed and non-empty) sets of worlds and times.

- Interpretations $F$ (according to models $M$) and assignments $g$:

– $F_M(\mathbf{c}) : (W \times T) \to D_\tau$, for any $\tau \in T_{IL}$ and $c \in Con_\tau$;

– $g(x) \in D_\tau$, for any $\tau \in T_{IL}$ and $x \in Var_\tau$.

(22) Denotations of *IL*-terms

a. $[\![\mathbf{c}]\!]^{M,(w_c,t_c),(w,t),g} = F(\mathbf{c})(w,t)$, if $\mathbf{c} \in Con_\tau$;
b. $[\![x]\!]^{M,(w_c,t_c),(w,t),g} = g(x)$ if $x \in Var_\tau$;
c. $[\![[\alpha = \beta]]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\alpha]\!]^{M,(w_c,t_c),(w,t),g} = [\![\beta]\!]^{M,(w_c,t_c),(w,t),g}$;
d. $[\![[\lambda x.\alpha]]\!]^{M,(w_c,t_c),(w,t),g}(u) = [\![\alpha]\!]^{M,(w_c,t_c),(w,t),g[x/u]}$, for any $u \in D_\sigma$ (where $x \in Var_\sigma$);
e. $[\![\alpha(\beta)]\!]^{M,(w_c,t_c),(w,t),g} = [\![\alpha]\!]^{M,(w_c,t_c),(w,t),g}([\![\beta]\!]^{M,(w_c,t_c),(w,t),g})$;
f. $[\![[{}^\wedge\alpha]]\!]^{M,(w_c,t_c),(w,t),g}(w',t') = [\![\alpha]\!]^{M,(w_c,t_c),(w',t'),g}$, for any $w' \in W$ and $t' \in T$;
g. $[\![[{}^\vee\alpha]]\!]^{M,(w_c,t_c),(w,t),g} = [\![\alpha]\!]^{M,(w_c,t_c),(w,t),g}(w,t)$.

(23) Accessibility relations and indexical operators (in intended models $M$)

a. $F_M(<)(w,t)(w',t') = 1$ iff $w = w'$ and $t' < t$;
b. $F_M(>)(w,t)(w',t') = 1$ iff $w = w'$ and $t < t'$;
c. $F_M(\sim_\mathbf{t})(w,t)(w',t') = 1$ iff $w = w'$;
d. $F_M(\sim_\mathbf{w})(w,t)(w',t') = 1$ iff $t = t'$;
e. $F_M(\mathbf{b})(u)(w,t)(w',t') = F_M(\mathbf{b})(u)(w,t)(w',t'')$ cf. Hintikka (1969)
f. $[\![\mathbf{A}\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{M,(w_c,t_c),(w_c,t),g} = 1$ cf. Kaplan (1979)
g. $[\![\mathbf{N}\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{M,(w_c,t_c),(w,t_c),g} = 1$ cf. Kamp (1971)

(24) Abbreviations

a. $(\forall x)\varphi$ is short for $[(\lambda x.\varphi) = (\lambda x.[x = x])]$
⇒ $[\![(\forall x)\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots,g[x/u]} = 1$, for all $u \in D_\sigma$
b. $\neg\varphi$ is short for $[\varphi = (\forall v)v]$;
⇒ $[\![\neg\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots} = 0$
c. $\varphi \wedge \psi$ is short for $(\forall f)[\varphi = [f(\varphi) = f(\psi)]]$;
⇒ $[\![\varphi \wedge \psi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots} = [\![\psi]\!]^{\cdots} = 1$
d. $\Box\varphi$ is short for $([{}^\wedge\varphi] = [{}^\wedge(\forall x)[x = x]])$
⇒ $[\![\Box\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots(w',t'),g} = 1$, for all $(w',t') \in W \times T$
e. $\mathbf{L}\varphi$ is short for $[\lambda p.\Box[[{}^\vee p] \rightarrow \varphi]](\sim_\mathbf{w})$
⇒ $[\![\mathbf{L}\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots(w',t),g} = 1$, for all $w' \in W$
f. $\mathbf{P}\varphi$ is short for $[\lambda p.\Diamond[[{}^\vee p] \wedge \varphi]](<)$
⇒ $[\![\mathbf{P}\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots(w,t'),g} = 1$, for some $t' < t$
g. $\mathbf{F}\varphi$ is short for $[\lambda p.\Diamond[[{}^\vee p] \wedge \varphi]](>)$
⇒ $[\![\mathbf{F}\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots(w,t'),g} = 1$, for some $t' > t$
h. $\mathbf{B}_\alpha\varphi$ is short for $[\lambda p.\Box[[{}^\vee p] \rightarrow \varphi]](\mathbf{b}(\alpha))$ where $\alpha \in IL_e$
⇒ $[\![\mathbf{B}_\alpha\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots(w',t),g} = 1$, for all $w'$ such that: $F_M(\mathbf{b})([\![\alpha]\!]^{M,(w_c,t_c),(w,t),g})(w,t)(w',t) = 1$
⇒ $[\![\mathbf{B}_\alpha\varphi]\!]^{M,(w_c,t_c),(w,t),g} = 1$ iff $[\![\varphi]\!]^{\cdots(w',t),g} = 1$, for all $w' \in Dox_{a,w,t}$ where $Dox_{u,w,t} = \{w' \in W | F_M(\mathbf{b})(u)(w,t)(w',t) = 1\}$ and $a = [\![\alpha]\!]^{M,(w_c,t_c),(w,t),g}$
i. $\exists$ and $\mathbf{H}$ are short for $\neg\forall\neg$ and $\neg\mathbf{P}\neg$, respectively
j. etc.

## 2.2 From *IL* to *Ty2*

(25) Standard translation of *IL* to *Ty2* cf. (Gallin, 1975, 61ff.)

a. $\overline{\mathbf{c}} = \mathbf{c}(i_0)$, for constants $\mathbf{c}$;
b. $\overline{x} = x$, for variables $x$;

c. $\overline{[\alpha = \beta]} = [\overline{\alpha} = \overline{\beta}]$;
d. $\overline{\alpha(\beta)} = \overline{\alpha}(\overline{\beta})$;
e. $\overline{[\lambda x.\alpha]} = [\lambda x.\overline{\alpha}]$;
f. $\overline{[{}^\wedge\alpha]} = [\lambda i_0.\overline{\alpha}]$;
g. $\overline{[{}^\vee\alpha]} = \overline{\alpha}(i_0)$.
h. $\overline{\mathbf{A}\varphi} = (\exists i_1)[\sim_{\mathbf{w}}(i_1)(i_0) \wedge \sim_{\mathbf{t}}(i_1)(c) \wedge [\lambda i_0.\overline{\varphi}](i_1)]$;
i. $\overline{\mathbf{N}\varphi} = (\exists i_1)[\sim_{\mathbf{w}}(i_1)(c) \wedge \sim_{\mathbf{t}}(i_1)(i_0) \wedge [\lambda i_0.\overline{\varphi}](i_1)]$

(26) *Lemma* (Gallin, 1975, 62)
Let $M$ be an *IL*-model, $g$ an appropriate assignment, $((w_c, t_c), (w, t))$ a point of reference, and $M^*$ and $g^*$ a *Ty2*-model and assignment where:

$$F_M \subseteq F_{M^*},\ g \subseteq g^*,\ g^*(c) = (w_c, t_c), \text{ and } g^*(i_0) = (w, t).$$

Then for any $\tau \in T_{IL}$ and $\alpha \in IL_\tau$:

$$[\![\alpha]\!]^{M,(w_c,t_c),(w,t),g} = [\![\overline{\alpha}]\!]^{M^*,g^*}.$$

## 2.3 Examples revisited

(3) $\Diamond(\forall x)[\reflectbox{$\mathbf{B}$}_0(Q(x)) \rightarrow Q(x)]$

(27) $(\exists i_1)[\sim_{\mathbf{w}}(i_1)(i_0) \wedge (\forall x)[Q_{i_0}(x) \rightarrow Q_{i_1}(x)]]$

(28) $[\lambda\mathcal{R}.\Diamond(\forall x)[\mathcal{R}(x) \rightarrow Q(x)]](Q)$

(4) Every man who ever supported the Vietnam War will have to admit that now he believes that he was an idiot then.

(5a) $(\forall x)[\mathbf{M}_{w,t}(x) \rightarrow (\forall t' < t)[\mathbf{S}_{w,t'}(x) \rightarrow (\exists t'' > t)$
$(\forall w')[\mathbf{A}_{w,t''}(x)(w') \rightarrow (\forall w'')[\mathbf{B}_{w',t_c}(x)(w'') \rightarrow \mathbf{I}_{w'',t'}(x)]]]]$

(29) $[\lambda\mathcal{O}_2.[\lambda\mathcal{O}_1.(\forall x)[M(x) \rightarrow \mathcal{O}_1({}^\wedge S(x) \rightarrow$
$[\lambda q.\mathcal{O}_2({}^\wedge\mathbf{B}_x(\mathbf{N}(\mathbf{B}_x({}^\vee q \wedge I(x)))))](\sim_{\mathbf{t}}))]](\lambda p.\mathbf{H}{}^\vee p)](\lambda q.\mathbf{F}{}^\vee q)$

(5a) $(\forall x)[\mathbf{M}_{w,t}(x) \rightarrow (\forall t' < t)[\mathbf{S}_{w,t'}(x) \rightarrow (\exists t'' > t')$
$(\forall w')[\mathbf{A}_{w,t''}(x)(w') \rightarrow (\forall w'')[\mathbf{B}_{w',t_c}(x)(w'') \rightarrow \mathbf{I}_{w'',t'}(x)]]]]$

(30) $[\lambda\mathcal{O}.(\forall x)[M(x) \rightarrow \mathcal{O}({}^\wedge S(x) \rightarrow$
$[\lambda q.\mathbf{FB}_x(\mathbf{N}(\mathbf{B}_x({}^\vee q \wedge I(x))))](\sim_{\mathbf{t}}))]]\ (\lambda p.\mathbf{H}{}^\vee p)$

(5a) $(\forall x)[\mathbf{M}_{w,t}(x) \rightarrow (\forall t' < t)[\mathbf{S}_{w,t'}(x) \rightarrow (\exists t'' > t_c)$
$(\forall w')[\mathbf{A}_{w,t''}(x)(w') \rightarrow (\forall w'')[\mathbf{B}_{w',t_c}(x)(w'') \rightarrow \mathbf{I}_{w'',t'}(x)]]]]$

(31) $[\lambda\mathcal{O}.(\forall x)[M(x) \rightarrow \mathcal{O}({}^\wedge S(x) \rightarrow$
$[\lambda q.\mathbf{NFB}_x(\mathbf{N}(\mathbf{B}_x({}^\vee q \wedge I(x))))](\sim_{\mathbf{t}}))]]\ (\lambda p.\mathbf{H}{}^\vee p)$

# 3 Adding Backwards-looking Operators to Intensional Type Logic

## 3.1 Implementing backwards-looking operators in *IL*

(32) *YIL* cf. Yanovich (2015)
For any type $\tau \in T_{IL}$
a. $[\![x]\!]^{M,(w_c,t_c),\rho,n,g} = g(x)$
b. $[\![\mathbf{c}]\!]^{M,(w_c,t_c),\rho,n,g} = F_M(\mathbf{c})(\rho(n))$
c. $[\![[\alpha = \beta]]\!]^{M,(w_c,t_c),\rho,n,g} = 1$ iff $[\![\alpha]\!]^{M,(w_c,t_c),\rho,n,g} = [\![\beta]\!]^{M,(w_c,t_c),\rho,n,g}$
d. $[\![\alpha(\beta)]\!]^{M,(w_c,t_c),\rho,n,g} = [\![\alpha]\!]^{M,(w_c,t_c),\rho,n,g}([\![\beta]\!]^{M,(w_c,t_c),\rho,n,g})$

e. $[\![[\lambda x.\alpha]]\!]^{M,(w_c,t_c),\rho,n,g}(u) = [\![\alpha]\!]^{M,(w_c,t_c),\rho,n,g[x/u]}$
f. $[\![[{}^\wedge\alpha]]\!]^{M,(w_c,t_c),\rho,n,g}(i) = [\![\alpha]\!]^{M,(w_c,t_c),\rho[n+1/i],n+1,g}$
g. $[\![[{}^\vee\alpha]]\!]^{M,(w_c,t_c),\rho,n,g} = [\![\alpha]\!]^{M,(w_c,t_c),\rho,n,g}(\rho(n))$
h. $[\![\mathbf{Ƀ}^l_r\varphi]\!]^{M,(w_c,t_c),\rho,n,g} = [\![\varphi]\!]^{M,(w_c,t_c),\rho[n+1/(\rho(l)_1,\rho(r)_2)],n+1,g}$
i. $[\![\mathbf{N}\varphi]\!]^{M,(w_c,t_c),\rho,n,g} = [\![\varphi]\!]^{M,(w_c,t_c),\rho[n+1/(\rho(n)_1,t_c)],n+1,g}$
j. $[\![\mathbf{A}\varphi]\!]^{M,(w_c,t_c),\rho,n,g} = [\![\varphi]\!]^{M,(w_c,t_c),\rho[n+1/(w_c,\rho(n)_2)],n+1,g}$

(33) a. $[\![\mathbf{Ƀ}^l_r\varphi]\!]^{M,(w_c,t_c),\rho,n,g} = [\![{}^\wedge\varphi]\!]^{M,(w_c,t_c),\rho,n,g}(\rho(l)_1,\rho(r)_2)$
b. $[\![\mathbf{N}\varphi]\!]^{M,(w_c,t_c),\rho,n,g} = [\![{}^\wedge\varphi]\!]^{M,(w_c,t_c),\rho,n,g}(\rho(n)_1,t_c)$
c. $[\![\mathbf{A}\varphi]\!]^{M,(w_c,t_c),\rho,n,g} = [\![{}^\wedge\varphi]\!]^{M,(w_c,t_c),\rho,n,g}(w_c,\rho(n)_2)$

(34) $(\forall x)[\mathbf{M}(x) \rightarrow \mathbf{H}(\mathbf{S}(x) \rightarrow \mathbf{Ƀ}^0_0(\mathbf{FB}_x(\mathbf{N}(\mathbf{B}_x(\mathbf{Ƀ}^6_1(\mathbf{I}(x)))))))]$ $(\equiv (29))$

(35) $(\forall x)[M(x) \rightarrow \mathbf{H}(S(x) \rightarrow \mathbf{FB}_x(\mathbf{N}(\mathbf{B}_x(\mathbf{Ƀ}^5_1(I(x))))))]$ $(\equiv (30))$

(36) $(\forall x)[M(x) \rightarrow \mathbf{H}(S(x) \rightarrow \mathbf{NFB}_x(\mathbf{N}(\mathbf{B}_x(\mathbf{Ƀ}^6_1(I(x))))))]$ $(\equiv (31))$

## 3.2 From *YIL* to *IL*

(37) *Theorem*
Let $M, M^*, g, g^*$ and $(w_c, t_c)$ be as in (26), and let $\rho$ be a sequence of indices (in $W \times T$) such that $\rho(k) = g^*(i_k)$ for any $k$.
a. If $\tau \in T_{IL}$, $\alpha \in YIL_\tau$, and $g^*(i_0) = g^*(c)$, then:
$[\![Z_0(\alpha)]\!]^{M^*,g^*} = [\![\alpha]\!]^{M,(w_c,t_c),\rho,0,g}$.
b. If $\varphi \in YIL_t$, then:
$[\![Z_0(\varphi)]\!]^{M^*,g^*} = [\![\varphi]\!]^{M,(w_c,t_c),\rho,0,g}$.

(38) *Corollary*
a. For any $\alpha \in YIL_\tau$ such that $\sigma(\alpha) = 0$, there is a $\gamma \in IL_\tau$ such that for any $M$, $(w_c, t_c)$, $g$, and $\rho$:
$[\![\alpha]\!]^{M,(w_c,t_c),\rho,0,g} = [\![\gamma]\!]^{M,(w_c,t_c),(w_c,t_c),g}$.
b. For any $\varphi \in YIL_t$ there is a $\psi \in IL_t$ such that for any $M$, $(w_c, t_c)$, $(w, t)$, $g$, and $\rho$:
$[\![\varphi]\!]^{M,(w_c,t_c),\rho,0,g} = [\![\psi]\!]^{M,(w_c,t_c),(w,t),g}$.

(39) *Translation*
a. $Z_n(x) = x$
b. $Z_n(\mathbf{c}) = \mathbf{c}(i_n)$
c. $Z_n([\alpha = \beta]) = [Z_n(\alpha) = Z_n(\beta)]$
d. $Z_n(\alpha(\beta)) = Z_n(\alpha)(Z_n(\beta))$
e. $Z_n([\lambda x.\alpha]) = (\lambda x.Z_n(\alpha))$
f. $Z_n([{}^\wedge\alpha]) = (\lambda i_{n+1}.Z_{n+1}(\alpha))$
g. $Z_n([{}^\vee\alpha]) = Z_n(\alpha)(i_n)$
h. $Z_n(\mathbf{Ƀ}^l_r\varphi) = [\lambda p.(\exists j)[\sim_{\mathbf{w}}(j)(i_l) \wedge \sim_{\mathbf{t}}(j)(i_r) \wedge p(j)]](Z_n({}^\wedge\varphi))$
$= [\lambda p.(\exists j)[\sim_{\mathbf{w}}(j)(i_l) \wedge \sim_{\mathbf{t}}(j)(i_r) \wedge p(j)]](\lambda i_{n+1}.Z_{n+1}(\varphi))$
i. $Z_n(\mathbf{N}\varphi) = [\lambda p.(\exists j)[\sim_{\mathbf{w}}(j)(i_n) \wedge \sim_{\mathbf{t}}(j)(c) \wedge p(j)]](\lambda i_{n+1}.Z_{n+1}(\varphi))$
j. $Z_n(\mathbf{A}\varphi) = [\lambda p.(\exists j)[\sim_{\mathbf{w}}(j)(c) \wedge \sim_{\mathbf{t}}(j)(i_n) \wedge p(j)]](\lambda i_{n+1}.Z_{n+1}(\varphi))$

(40) $Z_0(\alpha) \equiv \overline{\alpha}$, for any *IL*-term $\alpha$.

(41) *Lemma*
Let $M, M^*, g, g^*$ and $(w_c, t_c)$ be as in (26), and let $\rho$ be a sequence of indices (in $W \times T$) such that $\rho(k) = g^*(i_k)$ for any $k$. Then for any $\tau \in T_{IL}$, $\alpha \in YIL_\tau$ and natural number $n$:

$$[\![Z_n(\alpha)]\!]^{M^*,g^*} = [\![\alpha]\!]^{M,(w_c,t_c),\rho,n,g}.$$

(42) $[\![\alpha]\!]^{M,(w_c,t_c),\rho,0,g} = [\![\alpha]\!]^{M,(w_c,t_c),\rho(0),g}$, for any *IL*-term $\alpha$.

(43) *Theorem* (Zimmermann, 1989, 75)

If $\beta \in Ty2_\tau$ meets conditions a.–c., then there is a $\gamma \in IL_\tau$ such that $\overline{\gamma}$ is logically equivalent to $\beta$.

a. $\tau \in T_{IL}$;

b. if **c** is a constant occurring in $\beta$, then **c** is a constant of some type $\langle s, \sigma \rangle$, where $\sigma \in T_{IL}$;

c. if $x$ is a variable occurring freely in $\beta$, then $x \in Var_\sigma \cup \{i_0\}$ where $\sigma \in T_{IL}$.

# 4 Remaining issues

- Compositionality — Cresswell (1990), Schlenker (2006)
- Twisted senses — Zimmermann (2018)

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