On the Boolean Closure of Regular Languages

It is well known that regular languages are closed under Boolean operations. Standard textbooks prove the result via two separate constructions on finite state automata – one for conjunction, one for negation – and then refer to the equally well-known functional completeness of the latter (or to de Morgan's Laws, for restricted versions). The present note gives a simplified proof that generalizes and unifies the two constructions mentioned, thus avoiding reference to functional completeness.

Notation and Definitions
We will conceive of 1 and 0 as truth values, with 1 corresponding to truth. Moreover, we will refer to the truth value of a statement '…' using the notation '$\vdash \ldots \dashv$' ; hence '$\vdash \ldots \dashv$' is short for 'that truth value that is identical to 1 iff '…'. Given a natural number n, an *n-place truth table* is a function f with domain $\{0,1\}^n$ and values in $\{0,1\}$, i.e. $f: \{0,1\}^n \to \{0,1\}$. Standard examples include the one-place truth table of negation ($\neg$), and the two-place truth tables of conjunction ($\wedge$), disjunction ($\vee$), material implication, and material equivalence.

Given a number n and a non-empty set U, an *n-place Boolean operation on U* is a function F: $\wp(U)^n \to \wp(U)$ for which there is an n-place truth table f such that:

(*) $\quad F(X_1,\ldots,X_n) = \{x \in U \mid f(\vdash x \in X_1 \dashv, \ldots, \vdash x \in X_n \dashv) = 1\}$,

for any $X_1,\ldots, X_n \subseteq U$. Since F is determined by f in (*), we may and will write F as f^* (suppressing reference to U). Clearly, $\neg^*(X) = U \backslash X$; $\wedge^*(X,Y) = X \cap Y$; $\vee^*(X,Y) = X \cup Y$. The relation between truth tables and Boolean operations is well-known and well-studied.

Given a number n and a non-empty set U, an *n-place Boolean combination on U* is a function K: $\wp(U)^n \to \wp(U^n)$ for which there is an n-place truth table f such that:

(+) $\quad K(X_1,\ldots,X_n) = \{(x_1,\ldots,x_n) \in U^n \mid f(\vdash x_1 \in X_1 \dashv, \ldots, \vdash x_n \in X_n \dashv) = 1\}$,

for any $X_1,\ldots, X_n \subseteq U$. Since K is determined by f in (+), we may and will write F as f^+. Clearly, $\neg^+(X) = U \backslash X = \neg^*(X)$; $\wedge^+(X,Y) = X \times Y$; $\vee^+(X,Y) = (X \times U) \cup (U \times Y)$. The connection between conjunction and Cartesian products has been exploited in variable-free logic (cf. Bernays (1957)).

As usual, finite state automata M will be represented by quintuples $M = (Q,\Sigma,\delta,s,F)$, where Q are M's (finitely many) states, Σ is M's alphabet, δ: $Q \times \Sigma \to Q$ is M's transition function, $s \in Q$ is M's initial state, and $F \subseteq Q$ is the set of M's acceptance states. Also, M's extened transition function $\hat{\delta}: Q \times \Sigma^* \to Q$ is defined as usual; and so is the language $L(M)$ accepted by M.

Theorem
Let $M_1 = (Q_1,\Sigma,\delta_1,s_1,F_1)$,…, $M_n = (Q_n,\Sigma,\delta_n,s_n,F_n)$ be (deterministic) finite state automata (over the same alphabet Σ) and f an n-place truth table. Then there is a finite state automaton $M_f = (Q_f,\Sigma,\delta_f,s_f,F_f)$ such that:

$$L(M_f) = f^*(L(M_1),\ldots L(M_n)).$$

Proof
We put:
(1) $Q_f = Q_1 \times \ldots \times Q_n$
(2) $\delta_f((q_1,, \ldots ,q_n),a) = (\delta_1(q_1,a),\ldots,\delta_n(q_n,a))$, if $(q_1,, \ldots ,q_n) \in Q_f$
(3) $s_f = (s_1,\ldots,s_n)$
(4) $F_f = f^+(F_1,\ldots F_n)$
It is easily verified that this construction coincides with the standard ones if $f = \neg$ or $f = \wedge$. By induction on x's length $|x|$, we first show:
(5) $\widehat{\delta_f}((q_1, \ldots, q_n), x) = (\widehat{\delta_1}(q_1, x), \ldots, \widehat{\delta_n}(q_n, x))$,
for any $q_1 \in Q_1,\ldots,q_n \in Q_n$ and $x \in \Sigma^*$. The proof is exactly as in textbook treatments of intersection:

<u>Case 1</u>: $|x| = 0$:

	$\widehat{\delta_f}((q_1, \ldots, q_n), x)$	
=	$\widehat{\delta_f}((q_1, \ldots, q_n), \varepsilon)$	for $x = \varepsilon$, since $|x| = 0$
=	$(q_1, \ldots, q_n)$	by def. of $\widehat{\delta_f}$
=	$(\widehat{\delta_1}(q_1, \varepsilon), \ldots, \widehat{\delta_n}(q_n, \varepsilon))$	by def. of $\widehat{\delta_1}, \ldots, \widehat{\delta_n}$
=	$(\widehat{\delta_1}(q_1, x), \ldots, \widehat{\delta_n}(q_n, x))$	since $x = \varepsilon$

<u>Case 2</u>: $|x| = m + 1$
… and thus $x = ya$, for some $a \in \Sigma$ and y such that $|y| = m$. The induction gives us:
(I.H.) $\widehat{\delta_f}((q_1, \ldots, q_n), y) = (\widehat{\delta_1}(q_1, x), \ldots, \widehat{\delta_n}(q_n, y))$
and we may reason as follows:

	$\widehat{\delta_f}((q_1, \ldots, q_n), x)$	
=	$\widehat{\delta_f}((q_1, \ldots, q_n), ya)$	since $x = ya$
=	$\delta_f(\widehat{\delta_f}((q_1, \ldots, q_n), y), a)$	by def. of $\widehat{\delta_f}$
=	$\delta_f((\widehat{\delta_1}(q_{1,} y), \ldots, \widehat{\delta_n}(q_n, y)), a)$	by I.H.: $|y| = m$
=	$(\widehat{\delta_1}(q_{1,} ya), \ldots, \widehat{\delta_n}(q_n, ya))$	by (2)
=	$(\widehat{\delta_1}(q_{1,} x), \ldots, \widehat{\delta_n}(q_n, x))$	since $ya = x$

Now (5) puts us in a position to prove the theorem as follows:

	$x \in L(M_f)$	
iff	$\widehat{\delta_f}(s_f, x) \in F_f$	By def. of $L(M)$
iff	$\widehat{\delta_f}((s_1, \ldots, s_n), x) \in F_f$	by (3)
iff	$\widehat{\delta_f}((s_1, \ldots, s_n), x) \in f^+(F_{1,} \ldots F_n)$	by (4)
iff	$(\widehat{\delta_1}(s_1, x), \ldots, \widehat{\delta_n}(s_n, x)) \in f^+(F_{1,} \ldots F_n)$	by (5)
iff	$f(\vdash \widehat{\delta_1}(s_{1,} x) \in F_1 \dashv, \ldots, \vdash \widehat{\delta_n}(s_n, x) \in F_n \dashv) = 1$	by (+)
iff	$f(\vdash x \in L(M_1) \dashv, \ldots, \vdash x \in L(M_n)) = 1$	by def. of $L(M)$
iff	$x \in f^*(L(M_1), \ldots, L(M_n))$	by (*)

Discussion
The above proof reveals that Cartesian products play two independent rôles in the product construction used to prove closure under intersection: the product

on the entire sets of states ensures a parallel simulation of the two automata combined; the product on the acceptance states takes care of the Boolean relation of intersection.

Reference

Bernays, Paul: ‘Über eine natürliche Erweiterung des Relationenkalkuls’. In: A. Heyting (ed.), *Constructivity in Mathematics*. Amsterdam 1957. 1–14.