EXTENSIONS AND PROJECTIONS IN DEONTIC DEFAULT LOGIC

ANDRÉ FUHRMANN

Abstract. It will be argued that John Horty’s proposal for deontic default logics does not extend beyond very simple default theories without losing its intended interpretation. The principal impediment can be removed by basing default inference on projections rather than extensions.

Keywords. Default logic, deontic logic, imperatives, obligations, conditional obligations, default priorities.

1. Introduction

This is a brief report summarising some findings concerning John Horty’s [8] (see also bibliography therein) [9] proposal for deontic default logics. It will be argued that Horty’s proposal does not extend beyond very simple default theories without jeopardising the intended deontic interpretation. The principal problem is what will be described in Section 3 as the Reflexivity Problem. For a more detailed exposition of the problem and possible responses see Fuhrmann [4]. In Section 4 I shall make a proposal so as to bypass the Reflexivity Problem. The solution consists in substituting the usual inference relation of Default Logic, defined in terms of extensions, by an inference relation based on quantification over a slightly different family of sets (projections). We shall see that the Reflexivity Problem can indeed not arise and, in Section 5, that the intended interpretation can be maintained once we move to more complex default theories, including those that contain information as to the order in which defaults should be considered.
2. Background

Let FML be the set of formulae of some propositional language containing a complete set of Boolean connectives. A (simple) default relation \( D \) is a subset of \( FML^2 \). Where \( a \) and \( b \) are formulae, we also write \( a \Rightarrow b \) for \((a, b) \in D\) and call \( a \Rightarrow b \) a default. A default theory, in the sense of Reiter [11] and others, is a pair \((D, A)\), where \( D \) is a default relation, i.e. a set of defaults, and \( A \) is a subset of FML, a set of assumptions.

Given a default theory \((D, A)\), its set of assumptions \( A \) can be extended by “drawing on” defaults in \( D \). The idea is to look for defaults whose premiss is in \( A \) and then include the conclusion in the extended set of assumptions \( A' \), if the result is consistent. Then use \( A' \) to detach further default-conclusions, if this can consistently be done. Continue the process until the defaults have saturated the assumptions, i.e. until the set of assumptions can no longer be consistently extended in the above manner. A set of formulae that maximally extends \( A \) with the aid of \( D \) is called an extension of \((D, A)\).

In Section 4 we shall briefly recall a precise definition of the notion of an extension. Here it suffices to take note of two immediate and well-known consequences of the basic idea as sketched above. First, the result of the extension process may depend on the order in which defaults are considered for the purpose of detachment. To illustrate with a small example, let \((D = \{a \Rightarrow b, a \Rightarrow \neg b\}, A = \{a\})\). Now start with \(a \Rightarrow b\). Then \(b\) will be detached and the second default cannot consistently be used. If we start with \(a \Rightarrow \neg b\) instead, then \(\neg b\) will be detached and the first default cannot be used. So two different and mutually inconsistent extensions will be produced, depending on the order in which the defaults are used. Given a default theory, its set of extensions \(Ext(D, A)\) is in general not a singleton set. The set \(Ext(D, A)\) can be reduced by forcing the order in which defaults are to be activated. In this case we think of \(D\) as structured by some ordering \(\prec\), thus working with prioritised default theories, usually represented by triples \((D, A, \prec)\).

Second, an extension \(E\) of a default theory \((D, A)\) is an extension of its assumption set \(A\). So we have quite trivially

\[
A \subseteq E.
\]

Even only with the rough sketch above at hand we can say a bit more about extensions. Clearly, they are bound below by \(A\). In the lucky case that all defaults can be used for the purpose of extending \(A\), an extension will conjoin \(A\) with the set \(Concl(D) = \{b : a \Rightarrow b \in D\}\) of all conclusions of \(D\). Given that we wish to extract as much information as possible from \(A\) together with \(D\), the upper limit of an extension must be \(\text{Cn}(A \cup Concl(D))\), the closure of \(A \cup Concl(D)\) under logical consequence. In general then, we invariably have

\[
E = \text{Cn}(A \cup Concl(D')),
\]

for some \(D' \subseteq D\). (Variation comes in by choosing constraints that determine \(D'\).)
What can we infer from a set of assumptions $A$ given a set $D$ of defaults? Default Logic (DL) proposes to look at answers in which we quantify over $\text{Ext}(D,A)$. Two answers are salient:

(i) $A \vdash^d_D x$ iff $x$ is in every extension of $(D,A)$ — skeptical, cautious, disjunctive mode of inference (the definition usually found in the DL-literature);

(ii) $A \vdash^c_D x$ iff $x$ is in some extension of $(D,A)$ — credulous, brave, conflict mode of inference.

Note that in virtue of Inclusion both these “modes” of default inference satisfy Reflexivity $\quad A \vdash^d_D x, \forall x \in A.$

Unless noted otherwise, we shall focus on default inference in the sense of (i). We may therefore drop the superscript $d$ and think of $\vdash$ as standing for default inference in the disjunctive mode.

3. Horty’s default evaluation rule

Where $a$ is formula, we may represent the (unconditional) imperative “See to it that $a$ be the case!” by $!a$. Let $I$ be a set of such imperatives. Now suppose that we generate a set of defaults from $I$ such that

$\top \Rightarrow x \in D(I)$ iff $\exists x \in I$

There is a straightforward sense in which, according to $I$, it ought to be that $x$ just in case $\emptyset \vdash^c_{D(I)} x$. We may also say: the default theory $(D(I),\emptyset)$ supports the assertion that it ought to be that $a$ just in case $a$ can be derived ($\vdash$) by using the defaults in $D(I)$. Thus, if $D = D(I)$ and $A = \emptyset$, then we may give the predicate $\phi$ a deontic interpretation in the following biconditional:

$\text{(DER)} \quad (D,A) \text{ supports } \phi x \text{ iff } A \vdash^c_D x.$

This is, in effect, the default evaluation rule for deontic modals used in Horty [8]. It transports into the framework of DL an idea that goes back to van Fraassen [2, 3].

If we apply (DER) to default theories in which all defaults are of the form $\top \Rightarrow x$ and in which there are no assumptions, then the extensions will be the maximally consistent subsets of $\text{Concl}(D)$. To briefly return to the two modes of inference, if there are conflicting extensions, say one containing $a$, the other containing $\neg a$, then in the disjunctive (skeptical) mode, neither $a$ nor $\neg a$ enjoys the property $\phi$. If, on the other hand, we use the conflict (credulous) mode, then we have conflicting oughts, i.e. both $\phi a$ and $\phi \neg a$.

In Horty’s work such simple default theories are designed as mere stepping stones. They illustrate the basic idea to be preserved when considering more interesting default theories. Thus, Horty writes (2014, p. 438):

---

1 Horty formulates the rule differently, using $\vdash$ for the relation of support (and $\circ$ for $\phi$). But since support is defined in term of default inference, and since we have reserved $\vdash$ for the latter, we better not overload $\vdash$ by using it also for the former. Note also that without further ado and in contrast to the ought-operator in modal deontic logic the predicate $\phi$ is not iterable.
These [simple imperative default theories] are very simple, of course, but the normative interpretation can be generalised to richer theories as well— theories of the form \((D, A, \prec)\) in which the hard information from \(A\) may not be empty, the defaults from \(D\) might have nontrivial premises, and there might be real priority relations among them.

It is difficult, however, to see how this generalisation, preserving the initial idea, should be possible. The difficulties are detailed in Fuhrmann (2017). The principal stumbling block is as follows.

Consider the simplest case of a default theory with non-trivial default premises and a non-empty assumption set: \((D = \{a \Rightarrow b\}, A = \{a\})\). By Reflexivity, we have \(A \vdash_D a\), whence \(\phi a\) by the evaluation rule (DER). But \(a\) is a fact-stating assumption, or so we may suppose. So \(\phi\) cannot carry a deontic interpretation—unless we declare factual assumptions to be obligatory. The argument can be sidestepped by restricting \(A\) to assumptions generated from imperatives. But this is not really an option, for defaults could then only be triggered if they are generated from conditional imperatives in which the condition happens to be commanded. Though examples of such coincidences can be made up, the approach would be of little interest if it required that conditions and commands always coincide in this way. Call this train of thought the Reflexivity Problem.

The problem is aggravated when we try to implement priorities among defaults in a more flexible way than the one briefly mentioned above. Instead of fixedly structuring \(D\) by an order \(\prec\), we can incorporate priority information in the assumption set and let such priorities occur as premises or conclusions of defaults. In this way priorities can themselves be inferred by default inference. This requires an extension of the language by ordering propositions \(d \prec d'\), where \(d\) and \(d'\) name defaults. Since all ordering propositions now occur in \(A\), they enjoy deontic status (by (DER) & Reflexivity). But the ordering of defaults is just meant to help determining one’s obligations—the ordering is not itself obligatory, not in general anyway.

4. Projections versus extensions

If we are to generalise Horty’s approach beyond toy default theories of merely heuristic value, we need to solve the Reflexivity Problem. For, in the presence of Reflexivity for \(\vdash\) the evaluation rule (DER) overgenerates obligations. So we need to filter out the overgenerated items.

A first, simple approach to solve the problem is to supplement the right-hand-side of (DER) by a clause that aims at taking out the overgenerated items:

\[(†) \quad (D(I), A) \text{ supports } Ox \text{ iff } A \vDash_{D(I)} x \text{ unless } A \vdash x.\]

Now we can populate \(A\) with factual assumptions without having \(Oa\) for all \(A \vdash a\). We can also use these assumptions to trigger real defaults \(a \Rightarrow b\) as generated from conditional imperatives of the form \(!b/a\). Categorical obligations are encoded as before by defaults of the form \(\top \Rightarrow b\). Moreover, since \(A \vdash \top\) (for any \(A\)), we have the welcome side-effect that logical truths
are never obligatory. (By contrast, the somewhat strange $O^\top$ cannot be avoided in standard deontic logic where $O$ is treated as a normal modal operator.)

Since we work under the assumption that defaults are the only deontically loaded items in the theory $(D,A)$, nothing that can be inferred without their participation can enjoy deontic status. The unless-clause in (†) has the effect of filtering out propositions that can be inferred without triggering defaults. So we may expect that this clause removes the undesired items. Although this much is true, (†) overshoots the mark as the following example shows.

The default theory $(D = \{a \Rightarrow b\}, A = \{a \rightarrow b,a\})$ has only one extension $E = \text{Cn}(a \rightarrow b,a)$. On the one hand, since $b$ can be inferred from $A$ alone, $Ob$ is not supported according to (†). On the other hand, given that there is a bijection between the underlying set $I$ of imperatives and the defaults in $D$, the presence of $a \Rightarrow b$ in $D$ implies that there is a conditional imperative $!b/a$ in $I$. Since the condition $a$ obtains, the default $a \Rightarrow b$ can be triggered, giving $b$. So we should expect that $Ob$ is supported by the theory—contrary to what (†) rules. We have thus found that (†) undergenerates! We need a better way of solving the Reflexivity Problem.

The idea of the following proposal is simple. Defaults $a \Rightarrow b$ represent conditional imperatives. So only the conclusions of defaults should fall into the scope of the derivative ought-predicate, and no conclusion of a triggered default should be left out. Thus, we are looking for a function that partially projects a set of defaults, i.e. of pairs $(x,y)$, to their right-hand elements, $y$. The input to such a projection is, apart from $D$, the set $A$ of assumptions which trigger defaults and an ordering in which the defaults are to be considered for triggering. We here implement this idea by adjusting the inductive definition of an extension in DL as first proposed by Brewka [1] (see also Makinson [10]).

Let $(D,A)$ be a default theory and let $(D,<)$ be a strict total order of the defaults. Since we assume $D$ to be countable, we may think of such ordering as an indexing of the defaults by the natural numbers in their natural sequence. We start the construction of the projection of $D$ by $A$ under $<$ by putting

$$A_0 = \emptyset.$$  

(In the definition of extensions we would start with $A_0 = \text{Cn}(A)$ instead.)

In the step $A_{k+1}$ we look for the first default $x \Rightarrow y$ in $(D,<)$ such that (i) $y \not\in A_k$, (ii) $A \vdash x$, and (iii) $A_k \nvdash \neg y$. If there is such a default, then we put

$$A_{k+1} = \text{Cn}(A_k \cup \{y\});$$

otherwise we let $A_{k+1} = A_k$, thereby ending the construction. (In the definition of extensions we would replace (ii) $A \vdash x$ by (ii) $x \in A_k$.)

Finally we sum up:

$$A_{(D,<)} = \bigcup\{A_i : 0 \leq i \leq \omega\}.$$  

$P$ is a projection of $D$ by $A$ iff $P = \text{Cn}(A_{(D,<)})$, for some default ordering $(D,<)$.

[5]
The process of constructing a projection is just as cumulative as is the construction of extensions, i.e. in both cases we have

\[ A_0 \subseteq A_1 \subseteq \cdots. \]

But unlike in the case of extensions, default-conclusions detached at one stage in the construction cannot be used as premises to trigger defaults in later stages. This is as it should be, since default-conclusions represent imperatives, not factual assumptions that could be used to match the hypothesis of a hypothetical imperative. Factual assumptions reside only in the set \( A \), as reflected above in the condition (ii).

About extensions of \( A \) by \( D \) recall that

\[ E = \text{Cn}(A \cup \text{Concl}(D')), \]

for some \( D' \subseteq D \) and \( \text{Concl}(D') = \{ b : a \Rightarrow b \in D' \} \). Thus, \( A \subseteq E \), which generates the Reflexivity Problem. About projections of \( D \) by \( A \) note that

\[ P = \text{Cn}(\text{Concl}(D')) , \text{ some } D' \subseteq D. \]

Thus, typically we do not have \( A \subseteq P \). We now replace extensions by projections in the definition of \( \vdash \) (in both modes) and thus define a new pair of relations \( \vdash^* \) as follows:

(i) \( A \vdash_D^{*d} x \) iff \( x \) is in every projection of \( (D, A) \) — the disjunctive mode;
(ii) \( A \vdash_D^{*c} x \) iff \( x \) is in some projection of \( (D, A) \) — the conflict mode of inference.

Finally, we propose a new evaluation rule (based on a set \( I \) of hypothetical imperatives) for the ought-predicate \( O \):

\[(\text{DER}^*) \quad (D(I), A) \text{ supports } Ox \text{ iff } A \vdash_{D(I)}^* x. \]

Since the ought-predicate is now determined by quantifying over projections, we know that \( (D, A) \text{ supports } Ox \) only if \( x \in \text{Cn}(\text{Concl}(D')) \) for some \( D' \subseteq D \). Thus overgeneration cannot arise: mere assumptions cannot gain ought-status. Assumptions can gain such status only if they also occur as conclusions of defaults—which is as it should be.

How does \( (\text{DER}^*) \) treat the undergeneration example above? In the example we have \( D = \{ b \Rightarrow a \} \), \( A = \{ a \Rightarrow b, a \} \) and \( (\vdash) \) does not deliver \( Ob \). Since there is only one default, there is only one (trivial) ordering to consider.

\[
\begin{align*}
A_0 &= \emptyset, \\
A_1 &= \{ b \}, \text{ since } a \Rightarrow b \in D, \ (i) \ b \notin A_0, \ (ii) \ A \vdash a, \text{ and } (iii) \ A_0 \not\vdash \neg b. \\
A_2 &= A_1, \text{ since no defaults apply.}
\end{align*}
\]
Thus \( P = Cn(b) \) is the only projection, whence \( A \models_D b \) (in both modes of inference). That is to say, \( Ob \) is supported by \((D,A)\) while \( Oa \) is not. (We note in passing that \( OT \) is also supported, by the logical closure of projections. So we lose the welcome side-effect of the otherwise less felicitous rule \( \{ \} \).)

Projections are the result of using assumptions so as to detach the conclusions of default rules—to project the rule \((x,y)\) to \( y \). The interpretation of projections naturally depends on what we assume about the interpretation of defaults. Above we have followed Horty’s idea—inspired by van Fraassen—that defaults are intimately related to conditional imperatives: that there is a bijection between the two. But this is certainly not the dominant interpretation—better, perhaps: heuristic—considered in the literature. According to the standard interpretation of DL, defaults represent risky inference tickets. These are licences to proceed from premisses to conclusions with a caveat. Consequently, conclusions only reached by using such rules inherit the vulnerability of the rules used. Assumptions, by contrast, are treated as safe by hypothesis. Under this interpretation, a projection of \( A \) by \( D \) represents the risky information that can be extracted from \((D,A)\) given a fixed ordering of the defaults; an extension, by contrast, represents the total information, risky or safe, implicit in \((D,A)\) relative to an ordering of \( D \). Once the effects of particular orderings are cancelled out by quantification, we arrive at default inference in terms of projections. Under the interpretation at hand, such inference represents strictly risky inference. As far as one sometimes wishes to know whether information extracted from a default theory is safe from or vulnerable to an increase of assumptions, such a notion of inference in terms of projections can be useful.

We can sense a general idea at work in the last paragraph. Defaults can be seen as transforming into propositional rules certain conditional speech acts. (Recall that these need not be substantially conditional: the condition can be vacuous, i.e. \( \top \).) If the defaults are then applied to assumptions, we derive propositions which fall under a predicate that is obtained from the character of the acts considered. In the standard interpretation of DL, the act is that of assertion and the predicate is truth. In this case the evaluation rule (DER) is of little interest. For given that truth is redundant in the sense of the biconditional \( \text{true}(x) \iff x \), the rule just comes to this:

\[
(D, A) \text{ supports } x \iff A \models_D x.
\]

Things are different if we base \( D \) on a set of conditional commands. Suppose we treat commands in the Fregean manner, i.e. as applying a particular commanding “force” to a propositional content (Frege in “Der Gedanke”, 1918). We can isolate the propositional content and transform conditional commands into default rules which, in turn, can be applied to assumptions. A judiciously chosen evaluation rule—the proposal offered here is (DER*)—can then reveal those propositions that defeasibly enjoy ought-status on the basis of the commands issued and the assumptions made. The approach can be seen as solving, by brute regimentation, the Frege-Geach problem; cf. Geach [5, 6]. The same recipe can be applied to other Fregean forces:
wishes uttered, questions raised, damnations expressed, and so on. Here is an incomplete table of correspondences employable in this manner:

<table>
<thead>
<tr>
<th>Force</th>
<th>Act</th>
<th>Predicate φ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Assertive</td>
<td>Jack asserts that p</td>
<td>p: true</td>
</tr>
<tr>
<td>Imperative</td>
<td>Jack commands that p</td>
<td>p: ought to be</td>
</tr>
<tr>
<td>Optative</td>
<td>Jack wishes that p</td>
<td>p: wished for</td>
</tr>
<tr>
<td>Interrogative</td>
<td>Jack asks whether p</td>
<td>p: asked whether</td>
</tr>
<tr>
<td>Damnative</td>
<td>Jacks boohs p</td>
<td>p: damned</td>
</tr>
</tbody>
</table>

5. Conditional obligations and ordering defaults

**Conditional obligations.** Once the basic repair is done as above we can proceed to consider Horry’s evaluation rule for conditional obligations – but now with $\sim^*$ replacing $\sim$ in the original version:

$$(C^*) \quad (D, A) \text{ supports } O_y x \text{ iff } A, y \sim^*_D x$$

The original version overgenerated conditional oughts by the Reflexivity Problem: mere factual assumption automatically gained ought-status under arbitrary conditions. Thus, in particular, if $x, y \in A$, then we have both $O_y x$ and $O_x y$. This is now prevented by working with projections rather than extensions.

**Ordering defaults.** We here consider only the more interesting case of extending the language by priority propositions (PPs) of the form $d \prec d'$, where $d$ and $d'$ are names of defaults in $D$ and $\prec$ is a predicate applying to pairs of such names. For details as to how PPs are employed in DL see e.g. Makinson [10]. The basic idea is as follows. Let $(D, A)$ be a default theory possibly containing PPs as constituent formulae in $D$ or $A$. If we ignore what the PPs express, then the set $\text{Ext}(D, A)$ of extensions is determined by all default orderings $(D, <)$. If we take heed of the PPs, then certain orderings should be disconsidered. For example, suppose that we derive $d \prec d'$ from $A$, possibly using $D$, where $d$ and $d'$ name the defaults $\delta$ and $\delta'$ respectively. Then an extension in which we apply $\delta'$ before $\delta$ would disrespect the information $d \prec d'$, whence, it is not admissible. The inference relation $\sim$ should thus be defined in terms of admissible extensions.

Let us now consider the three places where PPs can occur: as assumptions; as premisses of defaults; as conclusions of defaults. First, assumptions. The evaluation rule (DER) in terms of extensions, suffering from the Reflexivity Problem, would give all PPs deontic status: they all ought to be the case. But this seems wrong. PPs should help us to determine what our obligations are, they are not in general themselves obligatory. The imperatives to which they can be taken to relate – “consider $\delta_1$ before $\delta_2$!” – are typically quite different from the imperatives we are interested in here. The latter are based on appraisals of acts; the former reflect the relative merits
of imperatives of that latter kind. The evaluation rule \((\text{DER}^*)\) in terms of projections keeps PPs out of the scope of the \(O\)-predicate—as long as they do not themselves reflect commands (see below). On the other hand, \((\text{DER}^*)\) allows to let PPs do the work they are designed to do in a way that is consonant with the deontic interpretation under which we here consider default theories. To take a very simple example, in the default theory

\[
D = \{\top \Rightarrow a, \top \Rightarrow \neg a\}, \quad A = \{(\top \Rightarrow a) \prec (\top \Rightarrow \neg a)\}
\]

we have encoded two conflicting commands together with information, in form of a PP, as to how the conflict should be resolved. Given the PP in \(A\), the theory has only one admissible extension, viz. \(\text{Cu}(a)\), whence \(Oa\) rather than \(O\neg a\) is supported in both the conflict and the disjunctive mode of inference \((\triangleright^*)\).

Next, PPs as premisses of defaults. These are of the form \((d_1 \prec d_2) \Rightarrow a\). We are assuming here that each default reflects a hypothetical imperative. We therefore need to drive the bijection between defaults and imperatives into the premisses of defaults if these happen to be PPs. Thus the default \((d_1 \prec d_2) \Rightarrow a\) corresponds to an imperative \(!a/i_1 \prec i_1\) where \(i_1\) is itself of the form \(!b/c\) (and likewise \(i_2\)). These are complicated imperative phrases but they do not sound confused: see to it that \(a\) given that you prefer the one command over the other.

Finally, PPs as conclusions of defaults, i.e. defaults of the form \(a \Rightarrow (d_1 \prec d_2)\), reflecting an imperative \(!i_1 \prec i_2)\)/\(a\) (with \(i_1\) and \(i_2\) further imperatives). The content of the command is an act of preference or choice, as in “better take the train than the car!” (“given that you are late”). So here we have a case where a default theory can support the subsumption of a PP under the ought-predicate.

6. Conclusion

I have argued that Hory’s project of a Deontic DL gets stuck right after the start. The principal problem is the inclusion property of extensions: Extensions contain the assumptions they extend. But these assumption typically do not have deontic status—they are no oughts. This is the Reflexivity Problem. The problem can be solved, if we move from extensions (of \(A\) by \(D\)) to projections (of \(D\) by \(A\)). The solution continues to support the intended interpretation if we move to considering an evaluation rule for conditional obligation or to default theories that include priority information.
References


ANDRÉ FUHRMANN
INSTITUT FÜR PHILOSOPHIE
GOETHE-UNIVERSITÄT
60629 FRANKFURT a. M.
fuhrmann@em.uni-frankfurt.de