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Short communication

A simple statistical model on the degree of interconnection in partially molten rocks

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Abstract. The connectiveness of the melt phase strongly influences physical properties of partially molten rock such as seismic absorption due to melt squirt or electrical conductivity. The interrelation between the connectiveness, the geometry and the fraction of melt is studied theoretically for the idealistic case of melt occurring in ellipsoidal (spheroidal or penny-shaped) inclusions. A critical melt fraction leading to complete grain-boundary wetting is found ranging between 2α and 5α, where α is the aspect ratio of the melt films. The probability of overlapping of randomly distributed inclusions is estimated by a statistical approach. This probability depends on the melt fraction β and the aspect ratio α and is used to define (a) the degree of interconnection and (b) the mean number n of connections to neighbouring inclusions. (a) may be important for seismic absorption due to melt squirt, while (b) controls the electrical conductivity. The function $n(\alpha, \beta)$ is given approximately by $n \simeq (5.65 + 1.72/\alpha)\beta$. It is concluded that in mantle regions of moderate seismic-velocity decrease a reduced degree of interconnection might be important if melt occurs either in films or pockets.

Key words: Partial melt – Degree of interconnection – Melt films

Introduction

Physical properties of partially molten rocks such as seismic velocity, absorption and electrical conductivity are strongly dependent on the melt fraction and the geometrical distribution of the melt phase. A lot of work has been spent on modelling of the physical properties for different melt geometries such as penny-shaped melt inclusions, melt films, tubes or pockets (Walsh, 1969; O'Connell and Budiansky, 1977; Mavko, 1980; Waff, 1974; Honkura, 1975; Chelidze, 1978; Shankland et al., 1981; Schmeling, 1983, 1985, 1986). Although few workers considered a variable degree of interconnection in their conductivity models (Waff, 1974; Shankland and Waff, 1974; Hermance, 1979), the interrelation between the melt fraction, the melt geometry and the degree of interconnection remained unexplored. Some indications can be obtained from laboratory work. While some experiments show that the degree of interconnection is high regardless of how small the fraction of melt in intergranular tubes is (Waff and Bulau, 1979), others find a decreasing degree of interconnection for small melt fractions in films (Arzi, 1978; Van der Molen and Paterson, 1979; Berckhemer et al., 1982).

The aim of the present paper is to develop a statistical model describing the degree of interconnection as a function of the melt fraction and melt geometry for the idealized case of ellipsoidal melt films or pockets. Such a model can be included in theories on the physical properties of partial molten rock as has been done by Schmeling (1983, 1985, 1986). The connectiveness of intergranular melt tubes will not be considered here.

Critical melt fractions for complete interconnection

In this section the critical melt fraction β_c for films leading to complete grain-boundary wetting will be estimated. Assume the rock to be composed of grains with plane surfaces. If a melt film occurs at the j-th face of the i-th grain, its aspect ratio can be defined by

$$\alpha_{ij} = c_{ij}/a_{ij} \tag{1}$$

where c_{ij} is the half thickness, a_{ij} the equivalent radius $(=\sqrt{F_{ij}/\pi})$ and F_{ij} the area of the film. The critical melt fraction is then given by

$$\beta_c = \frac{1}{\sqrt{\pi}} \frac{\sum_{i=1}^{M} \sum_{j=1}^{m_i} F_{ij}^{3/2} \alpha_{ij}}{\sum_{i=1}^{M} V_i} \tag{2}$$

where m_i is the number of faces of the i-th grain, M the total number of grains and V_i the volume of the i-th grain. Assuming no correlation between the aspect ratios and the film surface areas and between the shape and the volume of the grains, the arithmetic mean of the aspect ratios can be drawn in front of the summations and the sums can be evaluated giving

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$$\beta_c = \pi^{-1/2}\bar{\alpha}\bar{A} \tag{3}$$

where

$$A_i = \left(\sum_{j=1}^{m_i} F_{ij}^{3/2}\right) \Big/ V_i.$$

The bars indicate the arithmetic mean to be taken over all films or grains, respectively. A_i describes the shape of the grains ranging around 3.5 for compact polyhedrons like a truncated octahedron. A increases approximately to 5, 6 or 10 for octahedrons, cubes or tetrahedrons, respectively. With these values we can find bounds for β_c

$$2\alpha < \beta_c < 5\alpha. \tag{4}$$

The critical melt fraction as constructed above would lead to a loss of the shear strength of the composite material. O'Connell and Budiansky (1974, 1977) determined the shear modulus for a material containing fluid-filled cracks. The unrelaxed modulus (defined by assuming isolated inclusions) dropped to 0 at $\beta/\alpha = 5.9$, while the relaxed modulus (defined by assuming the same fluid pressure in all inclusions) vanished at $\beta/\alpha = 2.4$. This agrees well with (4) and strongly supports O'Connell and Budiansky's (1974, 1977) elastical analysis down to vanishing modulus, a point which was questioned by Bruner (1976).

Variable degree of interconnection

In this section a variable degree of interconnection is determined for statistically distributed and oriented melt films and spheroidal melt inclusions. Seismic absorption due to melt squirt requires fluid flow between neighbouring inclusions of different orientations (Mavko and Nur, 1975). On the other hand, electrical conductivity depends strongly on the existence of continuous liquid paths through the material. For the first problem, the degree of interconnection ζ will be defined by the probability of an average inclusion being connected or overlapping with at least one neighbouring inclusion. For the second problem, theory on random resistor networks shows that the mean number n of direct connections to neighbouring inclusions is the important quantity (Waff, 1974; Kirkpatrick, 1973; Schopper, 1966).

The quantities ζ and n are determined as follows. For simplicity only inclusions of same spheroidal shape and size are assumed. In the mathematical model statistically distributed and oriented inclusions are assumed. A possible configuration of two such inclusions with the half axes a and c (defining the aspect ratio $\alpha = c/a$) is illustrated in Fig. 1. If the total volume of the material is V_t and the total number of inclusions is N, the probability of finding i centres of neighbouring inclusions within the volume V around one inclusion denoted by the number 0 is given by

$$P = \left(\frac{V_t - V}{V_t}\right)^{N-i} \cdot \left(\frac{V}{V_t}\right)^i \cdot \binom{N}{i}. \tag{5}$$

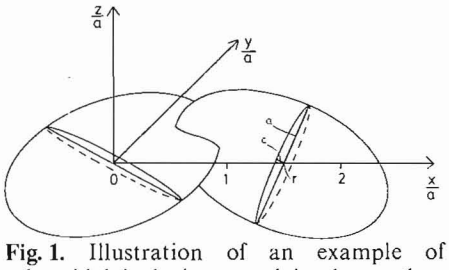

Fig. 1. Illustration of an example of two overlapping spheroidal inclusions used in the mathematical model. They have a distance r and an aspect ratio $\alpha = c/a$

For large N, P approaches the Poisson distribution

$$P = \exp(-V/v)\frac{V^i}{v^i}\frac{1}{i!} \tag{6}$$

where $v = V_i/\beta$ (V_i = inclusion volume) is the mean solid volume surrounding the inclusion 0. The probability of finding the centre of the $(i+1)$-th neighbouring inclusion in the additional volume dV is given by

$$\tilde{P} = \exp(-V/v)\frac{V^i}{v^i}\frac{1}{i!}\frac{dV}{v}. \tag{7}$$

Replacing v by $4\pi a^3\alpha/3\beta$ and V by $4\pi(r\cdot a)^3/3$, where r is the distance from the centre of the inclusion 0, normalized by the half axis a, the probability density $P_i dr$ of finding the centre of the i-th neighbouring inclusion in the interval $[r, r+dr]$ is given by

$$P_i dr = \exp\left(-\frac{r^3\beta}{\alpha}\right)\frac{1}{r}\left(\frac{r^3\beta}{\alpha}\right)^i \frac{3}{(i-1)!}dr. \tag{8}$$

In addition to the distances between neighbouring inclusions, the inclusion shapes will influence the probability of connection or overlapping. Assuming two inclusions with a distance r between their centre points (see Fig. 1), this probability P_r is calculated numerically for different aspect ratios for the range $0 < r < 2$ by integrating a function f (which equals 1 for the case of overlapping, else being 0) over all possible orientations of the two inclusions. The probability that the inclusion 0 overlaps with *any* of its neighbours (i.e. the degree of interconnection ζ) is finally given by

$$\zeta = 1 - \prod_{i=1}^{k}[1 - P_i'(\beta)] \tag{9}$$

where

$$P_i'(\beta) = \int_0^2 P_r(r)P_i(r,\beta)dr. \tag{10}$$

The product in Eq. (9) has to be extended over all k inclusions whose distance is smaller than 2.

The results for ζ as a function of melt fraction and aspect ratio are shown qualitatively in Fig. 2a and quantitatively in Fig. 2b. The curve for $\zeta = 99.5\%$ corresponds to melt fractions and aspect ratios leading to a vanishing shear modulus according to the film model of O'Connell and Budiansky (1977). For films, the melt fraction and aspect ratio can be combined by one pa-

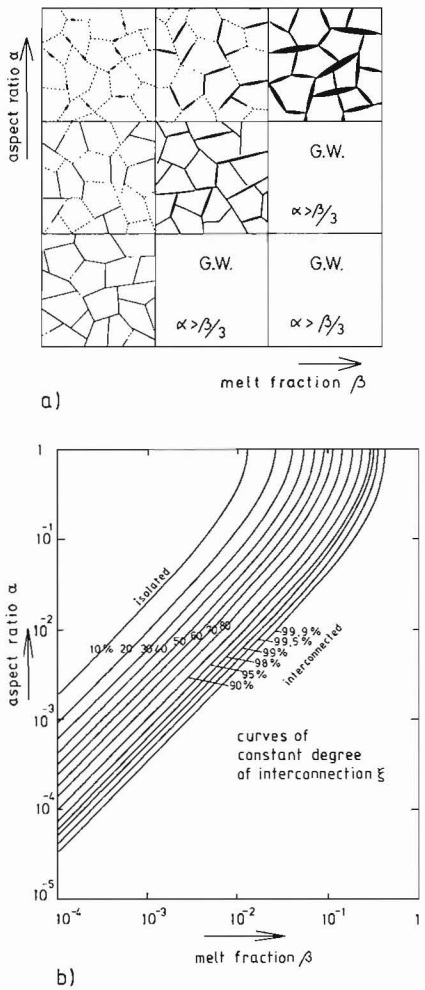

Fig. 2. **a** Illustration of the variable degree of interconnection for spheroidal inclusions as a function of melt fraction β and aspect ratio α. The fields *G.W.* correspond to melt fractions greater than β_c [see relation (4)] representing complete grain boundary wetting. **b** Curves of constant degree of interconnection ζ for statistically distributed geometrically similar spheroidal inclusions. The numbers beside the curves give ζ, the total probability of overlapping with neighbouring inclusions

rameter, the crack density ε (O'Connell and Budiansky, 1974)

$$\varepsilon = \frac{3}{4\pi}\frac{\beta}{\alpha}. \tag{11}$$

The degree of interconnection as a function of this crack density is shown in Fig. 3.

To determine the mean number of neighbouring inclusions with a direct connection to the considered inclusion, the probabilities P_i' [Eq. (10)] have to be summed over the k inclusions having the potential of overlapping at all or certain orientations

$$n(\beta) = \sum_{i=1}^{k} P_i'(\beta). \tag{12}$$

Figure (4) shows n as a function of α and β. As a comparison, the line $\alpha = \beta/3$ shows a possible critical

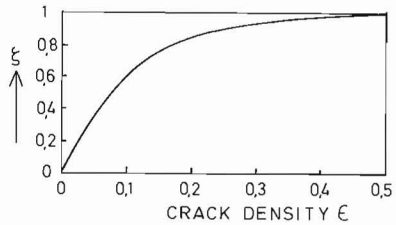

Fig. 3. Degree of interconnection ζ as a function of the crack density ε for films

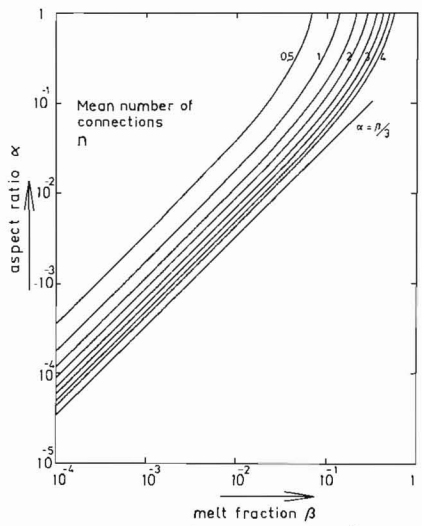

Fig. 4. Curves of constant n, the average number of connections (i.e. overlaps) with neighbouring inclusions for statistically distributed geometrically similar inclusions

melt fraction according to the relation (4), above which the loss of the shear strength has to be expected. The function $n(\alpha, \beta)$ can be approximated by a simple semi-empirical formula showing a standard deviation of about 2 %:

$$n(\alpha, \beta) \simeq (c_1 - c_2/\alpha)\beta \tag{13}$$

with $c_1 = 5.65$ and $c_2 = 1.72$.

Discussion and conclusion

As a main result it was found that the transition from isolated ($\zeta < 10\%$) to interconnected ($\zeta > 90\%$) inclusions covers more than a decade of the melt fraction (Fig. 2b) or a range from 0.01 to 0.25 in the crack density (Fig. 3). Such crack densities correspond to a decrease of the unrelaxed (relaxed) shear modulus of a few percent to about 25% (40%) (O'Connell and Budiansky, 1977). Thus, in regions of a moderate decrease of seismic velocities an incomplete degree of interconnection might be important if melt occurs in films or pockets. As was shown by Schmeling (1985, 1986), the seismic absorption due to melt squirt and the electrical conductivity would be lower than those determined e.g. by O'Connell and Budiansky (1977) and Shankland et al. (1981) who assumed complete interconnection.

The function $n(\alpha, \beta)$ (Fig. 4) has another important implication for the threshold probability of a least one

continuously connected liquid path through the material. Using percolation theory (Vyssotsky et al., 1961), Schmeling (1986) estimated this critical n-value to range between 0.75 and 1.5. This corresponds to a crack density between 0.1 and 0.2, implying that for smaller crack densities conduction is mainly controlled by the solid phase.

It should be emphasized that the model presented here is highly idealistic. It assumes ellipsoidal melt inclusions or films. Superpositions of different melt geometries were not considered. In nature, however, melt films are expected to develop by migration of melt from grain corners or edges into the grain faces. Furthermore, the system of grain faces in a polycrystalline material is principally different from a purely statistical distribution of surfaces. Nevertheless, by keeping these deficiencies of the model in mind the main principle conclusions should still be applicable to partially molten systems in which the melt occurs either in films or in melt pockets.

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